An Extremal Problem On Potentially $K_m - C_4$-graphic Sequences *

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Abstract

A sequence S is potentially $K_m - C_4$-graphical if it has a realization containing a $K_m - C_4$ as a subgraph. Let $\sigma(K_m - C_4, n)$ denote the smallest degree sum such that every n-term graphical sequence S with $\sigma(S) \geq \sigma(K_m - C_4, n)$ is potentially $K_m - C_4$-graphical. In this paper, we prove that $\sigma(K_m - C_4, n) \geq (2m-6)n - (m-3)(m-2) + 2$, for $n \geq m \geq 4$. We conjecture that equality holds for $n \geq m \geq 4$. We prove that this conjecture is true for $m = 5$.
Key words: graph; degree sequence; potentially $K_m - C_4$-graphic sequence
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1 Introduction

If $S = (d_1, d_2, ..., d_n)$ is a sequence of non-negative integers, then it is called graphical if there is a simple graph G of order n, whose degree sequence $(d(v_1), d(v_2), ..., d(v_n))$ is precisely S. If G is such a graph then G is said to realize S or be a realization of S. A graphical sequence S is potentially H-graphical if there is a realization of S containing H as a subgraph, while S is forcibly H-graphical if every realization of S contains H as a subgraph. Let $\sigma(S) = d(v_1) + d(v_2) + ... + d(v_n)$, and $[x]$ denote the largest integer less than or equal to x. We denote $G+H$ as the graph with $V(G+H) = V(G) \bigcup V(H)$ and $E(G+H) = E(G) \bigcup E(H) \bigcup \{xy : x \in V(G), y \in V(H)\}$. Let K_k,

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and C_k denote a complete graph on k vertices, and a cycle on k vertices, respectively. Let $K_m - C_4$ be the graph obtained from K_m by removing four edges of a 4 cycle C_4.

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turán number. This problem was proposed for $H = C_4$ by Erdös [2] in 1938 and in general by Turán [10]. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer l such that every n-term graphical sequence S with $\sigma(S) \geq l$ is forcibly H-graphical. Here we consider the following variant: determine the minimum even integer l such that every n-term graphical sequence S with $\sigma(S) \geq l$ is potentially H-graphical. We denote this minimum l by $\sigma(H, n)$. Erdös, Jacobson and Lehel [3] showed that $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$ and conjectured that equality holds. They proved that if S does not contain zero terms, this conjecture is true for $k = 3,\ n \geq 6$. The conjecture is confirmed in [4],[6],[7],[8] and [9].

Gould, Jacobson and Lehel [4] also proved that $\sigma(pK_2, n) = (p-1)(2n-2) + 2$ for $p \geq 2$; $\sigma(C_4, n) = 2[\frac{3n-1}{2}]$ for $n \geq 4$. Lai [5] proved that $\sigma(K_4 - e, n) = 2[\frac{3n-1}{2}]$ for $n \geq 7$. In this paper, we prove that $\sigma(K_m - C_4, n) \geq (2m-6)n - (m-3)(m-2) + 2$, for $n \geq m \geq 4$. We conjectured that equality holds for $n \geq m \geq 4$. We prove that this conjecture is true for $m = 5$.

2 Main results.

Theorem 1. $\sigma(K_m - C_4, n) \geq (2m-6)n - (m-3)(m-2) + 2$, for $n \geq m \geq 4$.

Proof. Let

$$H = K_{m-3} + \overline{K_{n-m+3}}$$

Then H is a uniquely realization of $((n-1)^{m-3}, (m-3)^{n-m+3})$ and H clearly does not contain $K_m - C_4$. Thus

$$\sigma(K_m - C_4, n) \geq (m-3)(n-1) + (m-3)(n-m+3) + 2 = (2m-6)n - (m-3)(m-2) + 2.$$

Theorem 2. For $n \geq 5$, $\sigma(K_5 - C_4, n) = 4n - 4$.

Proof. By theorem 1, for $n \geq 5$, $\sigma(K_5 - C_4, n) \geq 4n - 4$. We need to show that if S is an n-term graphical sequence with $\sigma(S) \geq 4n - 4$, then there is a realization of S containing a $K_5 - C_4$. Let $d_1 \geq d_2 \geq \cdots \geq d_n$, and let G be a realization of S.

Case: $n = 5$, if a graph has size $q \geq 8$, then clearly it contains a $K_5 - C_4$, so that $\sigma(K_5 - C_4, 5) \leq 4n - 4$.

Case: $n = 6$. If $\sigma(S) = 20$, we first consider $d_6 \leq 2$. Let S' be the degree sequence of $G - v_6$, so $\sigma(S') \geq 20 - 2 \times 2 = 16$. Then, by induction S' has a

realization containing a $K_5 - C_4$. Therefore S has a realization containing a $K_5 - C_4$. Now we consider $d_6 \geq 3$. It is easy to see that $S = (5^1, 3^5)$ or $S = (4^2, 3^4)$. Obviously, each is potentially $K_5 - C_4$-graphic. Next, if $\sigma(S) = 22$ then it must be that $d_6 \leq 3$. Let $S^{'}$ be the degree sequence of $G - v_6$, so $\sigma(S^{'}) \geq 22 - 3 \times 2 = 16$. Then $S^{'}$ has a realization containing a $K_5 - C_4$. Therefore S has a realization containing a $K_5 - C_4$. Finally, suppose that $\sigma(S) \geq 24$. We first consider $d_6 \leq 4$. Let $S^{'}$ be the degree sequence of $G - v_6$, so $\sigma(S^{'}) \geq 24 - 2 \times 4 = 16$. Then $S^{'}$ has a realization containing a $K_5 - C_4$. Therefore S has a realization containing a $K_5 - C_4$. Now we consider $d_6 \geq 5$. It is easy to see that $S = (5^6)$. Obviously, (5^6) is potentially $K_5 - C_4$-graphic.

Case: $n = 7$. First we assume that $\sigma(S) = 24$. Suppose $d_7 \leq 2$ and let $S^{'}$ be the degree sequence of $G - v_7$, so $\sigma(S^{'}) \geq 24 - 2 \times 2 = 20$. Then $S^{'}$ has a realization containing a $K_5 - C_4$. Therefore S has a realization containing a $K_5 - C_4$. Now we assume that $d_7 \geq 3$. It is easy to see that S is one of $(6^1, 3^6)$, $(5^1, 4^1, 3^5)$, or $(4^3, 3^4)$. Obviously, all of them are potentially $K_5 - C_4$-graphic. Next, if $\sigma(S) = 26$, It is easy to see that $d_7 \leq 3$. Let $S^{'}$ be the degree sequence of $G - v_7$, so $\sigma(S^{'}) \geq 26 - 3 \times 2 = 20$. Then $S^{'}$ has a realization containing a $K_5 - C_4$. Therefore S has a realization containing a $K_5 - C_4$. Finally, suppose that $\sigma \geq 28$. If $d_7 \leq 4$. Let $S^{'}$ be the degree sequence of $G - v_7$, so $\sigma(S^{'}) \geq 28 - 2 \times 4 = 20$. Then $S^{'}$ has a realization containing a $K_5 - C_4$. Therefore S has a realization containing a $K_5 - C_4$. Now we consider $d_7 \geq 5$. It is easy to see that $\sigma(S) > 5 \times 7 = 35$. Clearly, $d_7 \leq 6$. Let $S^{'}$ be the degree sequence of $G - v_7$, so $\sigma(S^{'}) > 35 - 6 \times 2 = 23$. Then, by induction $S^{'}$ has a realization containing a $K_5 - C_4$. Therefore S has a realization containing a $K_5 - C_4$.

We proceed by induction on n. Take $n \geq 8$ and make the inductive assumption that for $7 \leq t < n$, whenever S_1 is a t-term graphical sequence such that

$$\sigma(S_1) \geq 4t - 4$$

then S_1 has a realization containing a $K_5 - C_4$. Let S be an n-term graphical sequence with $\sigma(S) \geq 4n - 4$. If $d_n \leq 2$, let $S^{'}$ be the degree sequence of $G - v_n$. Then $\sigma(S^{'}) \geq 4n - 4 - 2 \times 2 = 4(n-1) - 4$. By induction, $S^{'}$ has a realization containing a $K_5 - C_4$. Therefore S has a realization containing a $K_5 - C_4$. Hence, we may assume that $d_n \geq 3$. By Proposition 2 and Theorem 4 of [4] (or Theorem 3.3 of [6]) S has a realization containing a K_4. By Lemma 1 of [4] ,there is a realization G of S with v_1, v_2, v_3, v_4, the four vertices of highest degree containing a K_4. If $d(v_2) = 3$, then $4n - 4 \leq \sigma(S) \leq n - 1 + 3(n-1) = 4n - 4$. Hence, $S = ((n-1)^1, 3^{n-1})$. Obviously, $((n-1)^1, 3^{n-1})$ is potentially $K_5 - C_4$-graphic. Therefore, we may assume that $d(v_2) \geq 4$. Let v_1 be adjacent to v_2, v_3, v_4, y_1. If y_1 is adjacent to one of v_2, v_3, v_4, then G contains a $K_5 - C_4$. Hence, we may assume that

y_1 is not adjacent to v_2, v_3, v_4. Let v_2 be adjacent to v_1, v_3, v_4, y_2. If y_2 is adjacent to one of v_1, v_3, v_4, then G contains a $K_5 - C_4$. Hence, we may assume that y_2 is not adjacent to v_1, v_3, v_4. Since $d(y_1) \geq d_n \geq 3$, there is a new vertex y_3, such that $y_1y_3 \in E(G)$.

If $y_3v_1 \in E(G)$, then G contains a $K_5 - C_4$. Hence, we may assume that $y_3v_1 \notin E(G)$. Then the edge interchange that removes the edges y_1y_3, v_1v_4 and v_2y_2 and inserts the non-edges y_1v_2, y_3v_1 and y_2v_4 produces a realization $G^{'}$ of S containing a $K_5 - C_4$ where $K_5 - C_4$ with five vertices v_1, v_2, v_3, v_4, y_1.

This finishes the inductive step, and thus Theorem 2 is established.

We make the following conjecture:

Conjecture.

$$\sigma(K_m - C_4, n) = (2m-6)n - (m-3)(m-2) + 2$$

for $n \geq m \geq 4$.

This conjecture is true for $m = 4$, by Theorem 5 of [4], and for $m = 5$, by the above Theorem.

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